Aerodynamic Theory for a Cascade of Oscillating Airfoils in Subsonic Flow

W. P. Jones* and Jimmie A. Moore†
Texas A & M University, College Station, Texas

The theory is developed in terms of the velocity potential rather than the acceleration potential, and a brief outline is given of the simple numerical technique used. The effects of varying airfoil spacing, frequency, Mach number, stagger angle, and phase difference between adjacent airfoils are discussed. Particular attention is given to variations in the aerodynamic damping for pure vertical translational and pitching motions. It is shown that the translational damping can become zero at certain discrete frequencies but that it never becomes negative. The pitching moment damping, however, can become negative over a wide range of frequencies of practical interest. The airfoils are assumed to be at zero mean incidence.

Introduction

MANY research workers have published papers on the theory of incompressible flow through a cascade of oscillating airfoils representing the blades of an axial compressor, and on the airloads induced by oscillatory inlet flow, but not as many have taken compressibility effects into account. Much of the work done for incompressible flow conditions has been reviewed by Meister[1] and by Schorr and Ready.[2] References to the most significant papers are given and the different methods of solution used are briefly discussed. However, the present paper is principally concerned with compressibility effects, though the theory and techniques developed can also be applied to incompressible flow as a limiting case when $M \rightarrow 0$. The general numerical method of solution employed is extremely simple to use and it enables the research worker to obtain meaningful and accurate results without getting involved in the very complex mathematics required in the analytical treatment of the cascade problem.

Much of the earlier work[5-8] on airfoils in cascade has been done in relation to either in-phase oscillations or to the determination of wall interference effects on measurements of airloads on oscillating airfoils in wind tunnels. The latter problem has been considered by many authors for incompressible flow conditions, but the literature on the corresponding compressible flow problem is somewhat scanty. Runyan[9] and his co-workers at NASA Langley investigated the problem and showed that it was possible to get tunnel resonance at certain critical frequencies. In an independent study, W.P. Jones[10] determined the airloads on a particular airfoil describing pitching oscillations and compared his results with those measured in a subsonic wind tunnel after making allowance for wall effects. More recently, investigations of subsonic flow through cascades of airfoils have been made in relation to vibration problems of axial compressors. Fleeter,[11] for instance, has considered the case of oscillatory inlet flow on a row of staggered airfoils at various Mach numbers up to $M = 0.9$ for a range of frequencies and selected angles. The simple technique used in this paper has recently been applied to the same problem and the results obtained have been reported in Ref. 12. Good agreement was obtained in all cases with Fleeter's results.[11]

Received March 17, 1975; revision received January 9, 1976. Research sponsored by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Grant No. 74-2700.
Index categories: Nonsteady Aerodynamics; Airbreathing Propulsion, Subsonic.

*Emeritus Professor, Aerospace Engineering Dept. AIAA Fellow.
†Senior Research Systems Analyst, Data Processing Center.

The present paper is, however, concerned with the equally important problem of determining the airload coefficients for an individual blade of a cascade of oscillating blades or airfoils. It summarizes work previously described in Ref. 13 and gives a selection of the results included in that study in a revised form. To preserve generality, a typical row of compressor blades is represented as in Fig. 1, where the blades are assumed to be staggered, to be oscillating out of phase, and to be spaced apart at specified distances. In order to determine the airloads on a typical airfoil of such a cascade, a method similar to that developed in Ref. 14 was used. The integral equation that one has to solve is replaced by a set of simultaneous equations that can readily be solved to give the required information. In the analysis, the downwash induced at the reference airfoil by the doublet distribution over each oscillating blade of the cascade is represented by a complicated series of Hankel functions denoted by S_0. This series, in the form in which it initially arises, is poorly convergent. Fortunately, however, it has been shown in Refs. 15 and 16 that it can be replaced by a rapidly convergent series of exponential terms. By using the alternative exponential form of S_0 and its derivative $S_I\ (\equiv \partial S_0/\partial x)$ the problem can easily be solved.

It was found that all the airload coefficients varied rapidly with reduced frequency up to the first critical frequency. Under certain conditions, it was also discovered that the pitching moment damping could be negative. This suggests that, for

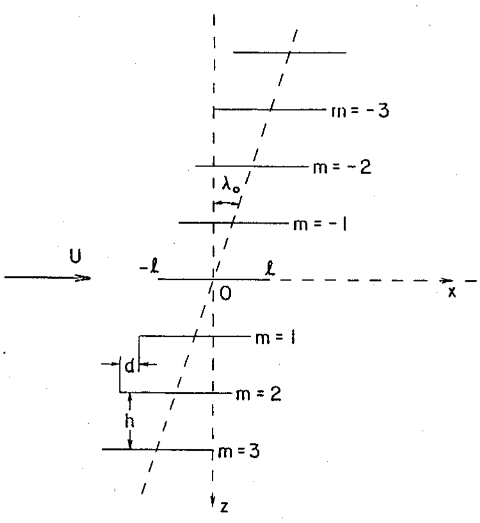


Fig. 1 Cascade of airfoils.

certain combinations of the relevant parameters, Mach number, frequency, spacing, stagger, etc., the airfoils might be unstable and that a one-degree-of-freedom type of oscillation could develop. It is also indicated that care must be taken to use the appropriate airload coefficients in any flutter study since they are very sensitive to some of the parameters involved.

Outline of Analysis

The analysis is developed for the cascade of oscillating airfoils illustrated in Fig. 1. The reference airfoil, with its center at the origin of coordinates, is assumed to be describing plunging and pitching oscillations of frequency p rad/sec defined by $z(=\ell z' e^{ipt})$ and $\alpha(=\alpha' e^{ipt})$ respectively. In general, it may also be assumed that the mth airfoil of the cascade has a phase lead of $m\sigma'$, where $\sigma'(=2\pi\delta')$ represents the difference in phase between adjoining airfoils. The distance between the mean positions of the leading edges of the airfoils is taken to be s and the stagger angle is assumed to be λ_0. The vertical spacing between the airfoils is then $h=s\cos\lambda_0$ and the lateral spacing is $d=s\sin\lambda_0$.

Let us next transform the coordinates into a nondimensional form by writing

$$x=\ell X, \quad z=\ell Z/\beta, \quad \text{and} \quad t=\ell T/U \tag{1}$$

where $\beta=(1-M^2)^{1/2}$. The downwash at any point X_1, 0 on the reference airfoil is then given by $w(X_1,T)\,(\equiv w'(X_1)\,e^{i\omega T})$ where

$$w'(X_1)=U[i\omega(z'+X_1\alpha')+\alpha'] \tag{2}$$

and $\omega=p\ell/U$ is the reduced frequency parameter. If the velocity potention ϕ of the flow is now replaced by a modified velocity potential Φ such that

$$\phi=U\ell\,\Phi e^{i(\lambda X+\omega T)} \tag{3}$$

where $\lambda=M^2\omega/\beta^2$, it may be deduced that

$$\partial^2\Phi/\partial X^2+\partial^2\Phi/\partial Z^2+\kappa^2\Phi=0 \tag{4}$$

where $\kappa=M\omega/\beta^2$. It also follows from Eq. (3) that the modified downwash distribution corresponding to $w'(X_1)$ of Eq. (2) is

$$W(X_1)=\frac{\partial\Phi}{\partial Z}=\frac{w'(X_1)e^{-i\lambda X_1}}{U\beta} \tag{5}$$

The corresponding formula for the local lift distribution $\ell(X)$ may also be expressed in the form

$$\ell(X)=\rho U^2(i\upsilon K+\partial K/\partial X)e^{i(\lambda X+\omega T)} \tag{6}$$

where $K(\equiv\Phi_a-\Phi_b)$ denotes the discontinuity in the modified velocity potential and the parameter $\upsilon=\omega/\beta^2$. The appropriate solution of Eq. (4), which satisfies the boundary condition defined by Eq. (5), may then be obtained from the integral equation

$$2\pi W(X_1,Z_1)=-\int_{-1}^{\infty}K(X)\,\frac{\partial^2}{\partial Z_1^2}\,S_0(X_1-X,Z_1,S,\lambda_M,\sigma)\,\mathrm{d}X \tag{7}$$

where

$$S_0=\frac{\pi i}{2}\sum_{m=-\infty}^{\infty}e^{im\sigma}H_0^{(2)}\{\kappa[(X_1-X+mD)^2+(mH-Z_1)^2]^{1/2}\} \tag{8}$$

$\sigma=\sigma'+\lambda D=2\pi\delta$ is the modified phase difference and Z_1,0 on the reference airfoil. It should be noted that σ is dependent on the actual phase shift σ' between adjacent blades, the Mach number, frequency, spacing, and stagger angle. The symbols $D(=d/\ell)$ and $H(=\beta h/\ell)$ in Eq. (8) correspond to $S\sin\lambda_M$ and $S\cos\lambda_M$ respectively where $\tan\lambda_M=\beta^{-1}\tan\lambda_o$ and $S=[D^2+H^2]^{1/2}$. In Ref. 16, the series of Hankel Functions defined by Eq. (8) is replaced by a series of exponential terms of the form

$$S_0=-\tfrac{1}{2}\sum_{n=-\infty}^{\infty}\frac{\exp[-2i\xi(\delta-n)\sin\lambda_M-2|\xi|[(\delta-n)^2-\mu^2]^{1/2}\cos\lambda_M]}{(\delta-n)^2-\mu^2]^{1/2}} \tag{9}$$

where $\xi=\pi(X_1-X)/S$ and $\mu=\kappa S/\pi$. Evidently, this series becomes divergent whenever

$$\mu=\delta,\ 1\pm\delta,\ \ldots,\ n\pm\delta,\ \ldots$$

or

$$\kappa S=\sigma'+\lambda D,\ 2\pi\pm(\sigma'+\lambda D),\ \text{etc}\ldots \tag{10}$$

and, at each of these critical values of μ (or κS), no solution is possible. The alternative form of Eq. (10) has also been given in Ref. 17 except that the sign of σ', the phase shift between the blades, is opposite to that assumed in this paper. From Eq. (10), the critical or "resonance" frequencies can readily be deduced.

In both papers listed as Ref. 13, the fact that $\sigma(=2\pi\delta)$ was the *modified* phase difference was overlooked. Consequently, Eq. (17) of the first paper and the curves in Fig. 2 of the second paper require correction. The critical reduced frequencies can be derived from Eq. (10) and are given by

$$\omega_c=(2\pi n+\sigma')[(1-M^2\cos^2\lambda_0)^{1/2}+M\sin\lambda_0]\ell/Ms \tag{11}$$

and

$$\omega_c=(2\pi n-\sigma')[(1-M^2\cos^2\lambda_0]\ell/Ms \tag{12}$$

where $n=0$, 1, 2, 3, etc., and only positive values of ω_c are

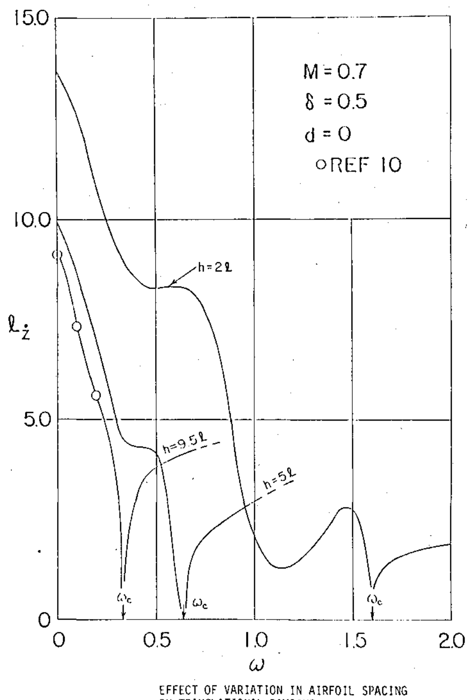


Fig. 2 Effect of variation in airfoil spacing on translational damping for $\lambda_0=0$.

considered. In Ref. 13 it is shown that Eq. (7) can be replaced by

$$2\pi W(X_I) = \int_{-I}^{I} K(X)\left[\frac{\partial S_I}{\partial X} + \kappa^2 S_0\right] dX + K_T I \qquad (13)$$

where K_T is the value of K at the trailing edge, $X=I$, and $S_I = \partial S_0/\partial X$. In the wake $K(X) = K_T \exp[-i\nu(X-I)]$, and it can be deduced that

$$I = -S_{IT} - i\nu S_{0T} - \nu^2(I-M^2)P \qquad (14)$$

where

$$P = -\tfrac{1}{2}\sum_{n=-\infty}^{\infty} \frac{S\exp[-2\pi\bar{a}(n)(I-X_I)/S]}{[2\pi\bar{a}(n)+i\nu S][(\delta-n)^2-\mu^2]^{1/2}} \qquad (15)$$

and $\bar{a}(n) = [(\delta-n)^2-\mu^2]^{1/2}\cos\lambda_M - i(\delta-n)\sin\lambda_M$. The terms S_{0T} and S_{IT} respectively denote the values of S_0 and S_I at the trailing edge. By dividing the reference airfoil into N equal strips of width $2b$ and assuming K to be constant over each strip, Eq. (13) can be reduced to a linear set of simultaneous equations when the boundary condition is satisfied at the center of each strip. The values of K_n ($n=1, 2, \ldots, N$) will then give the required K distribution. In Eq. (13), as suggested in Ref. 13, K_T is assumed to be given approximately by

$$K_T = \frac{K_N}{2i\nu b + \exp(-i\nu b)} \qquad (16)$$

where $2b$ is the width of each strip. This relation is derived by imposing the condition of no lift at the trailing edge and in the wake, namely $i\nu K + \partial K/\partial X = 0$, and expressing it in finite difference form.

Once the K_n values have been determined, the lift L $(=L'e^{ipt})$ and nose-up pitching moment $M(=M'e^{ipt})$ about half-chord can be deduced. In terms of aerodynamic coefficients, they are conveniently expressed as

$$L'/\rho U^2\ell = (\ell_z + i\omega\ell_{\dot z})z' + (\ell_\alpha + i\omega\ell_{\dot\alpha})\alpha' \qquad (17a)$$

and

$$M'/\rho U^2\ell^2 = (m_z + i\omega m_{\dot z})z' + (\omega_\alpha + i\omega m_{\dot\alpha})\alpha' \qquad (17b)$$

where the values of ℓ_z, $\ell_{\dot z}$ etc. can be calculated. The direct damping coefficients, $\ell_{\dot z}$, and $-m_{\dot\alpha}$, are of primary importance in stability studies. The effects of varying various parameters on these particular coefficients were investigated and some of the results obtained will next be discussed. When the reference axis is changed to quarter-chord, the coefficients in Eq. (17) are replaced by barred coefficients $\bar\ell_z$, $\bar\ell_{\dot z}$ etc.

Discussion of Results

In the two reports listed as Ref. 13, a considerable amount of detailed information on the behavior of the aerodynamic coefficients ℓ_z, $\ell_{\dot z}$ etc. is given. Such information would be required in any proposed study of the flutter characteristics of a cascade of airfoils but until now very little has been available. This lack of data is probably due to the fact that the methods of computation used, such as the Lane & Friedman method,[17] require substantial computing time and are prohibitively expensive as far as the systematic tabulation of aerodynamic coefficients is concerned.

Because of space limitations, only a few selected results from Ref. 13 are presented in this paper. Those given in Ref. 13 for staggered blades are given for constant values of σ rather than σ', the actual phase difference. The present paper contains some results for the latter case. Attention is principally concentrated on the damping aerodynamic coefficient, $\ell_{\dot z}$, for vertical translational oscillations and $-\bar m_{\dot\alpha}$, the pitching damping coefficient referred to the quarter-chord axis position. For instance, Fig. 2 shows the effect on $\ell_{\dot z}$ of varying the airfoil spacing when $M=0.7$, the airfoils are 180° out of phase, and $\lambda_0=0$. The main effect of increasing the spacing between the airfoils results in a decrease in the first critical frequency, ω_c, for which $\ell_{\dot z}=0$. As ω increases, the damping decreases rapidly to zero at $\omega=\omega_c$ and then increases again as ω is increased without becoming negative. Below the first critical frequency, the damping decreases when the spacing between the airfoils is increased. When $h=9.5\ell$, which corresponds to the case of an airfoil in a wind tunnel, the values of $\ell_{\dot z}$, check with results obtained previously by a different method.[10] The variation in $\ell_{\dot z}$ with phase difference and frequency is shown in Fig. 3 for the case when $M=0.7$, $h=2\ell$ and $\lambda_0=0$. For $0<\omega<\omega_c$, the damping decreases rapidly as ω is increased. A reduction in phase difference also results in a loss of damping for the lower values of ω. It should be noted that the first critical frequency, ω_c, is directly proportional to the phase difference. Figure 4 shows the effect of Mach number on $\ell_{\dot z}$ and the curves again reveal a rapid decrease in damping for the lower values of ω, particular for $M=0.9$. The variation in $\ell_{\dot z}$ when the phase difference is reduced from 180° (δ'0.5) is shown in Fig. 5 for the case of $M=0.9$ and a stagger angle of 60°. For the lower values of ω, a reduction in phase difference results in a reduction in translational damping. Similarly, in Fig. 6, it is shown that an increase in stagger angle results in a decrease in $\ell_{\dot z}$ when ω is small ($\omega \leq 0.1$).

Some typical results for the pitching moment damping coefficient $-\bar m_{\dot\alpha}$, referred to the quarter-chord axis are illustrated in Fig. 7-11. When $M=0.7$ and $\lambda_0=0$, the damping can be negative when the phase difference between the motions of the airfoils is greater than 60° ($\delta'=1/6$) as shown in Fig. 7. When $\delta'=\tfrac{1}{2}$, the damping is negative over the range $1.0<\omega\leq 1.6$, the upper value corresponding to the first "resonance" frequency. By decreasing the phase difference the region of potential instability is reduced until at $\delta'=1/6$, it almost disappears. The effect of Mach number is shown in Fig. 8, where it is revealed that, for the case considered, the region of negative damping is increased as Mach number is increased.

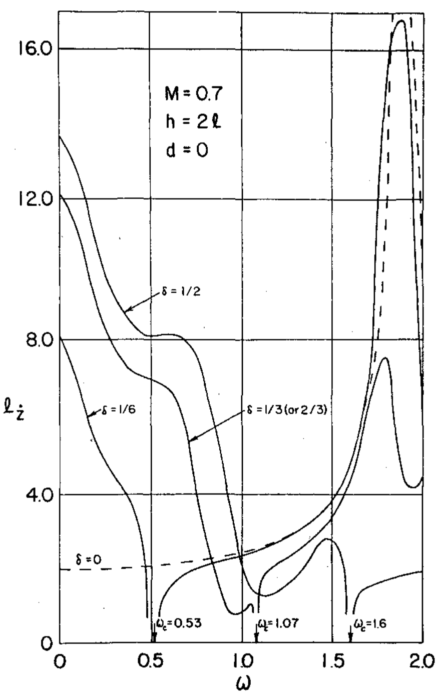


Fig. 3 Variation of translational damping with phase difference for $\lambda_0=0$.

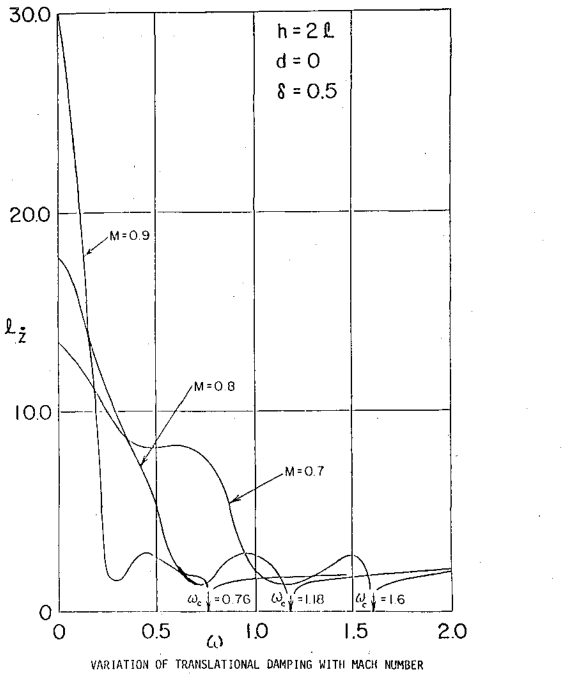


Fig. 4 Variation of translational damping with Mach number for $\lambda_0 = 0$.

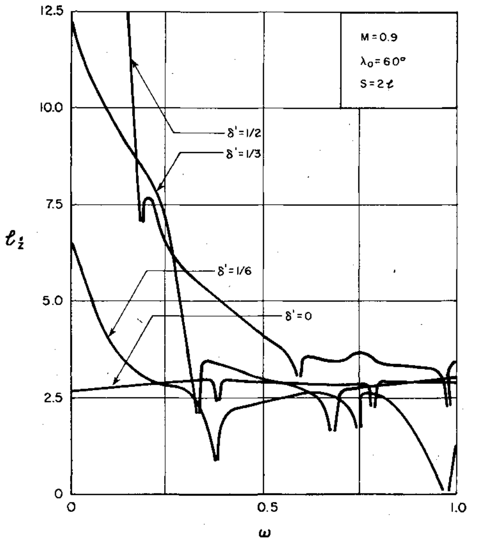


Fig. 5 Effect of variation in phase on translational damping when $\lambda_0 = 60°$.

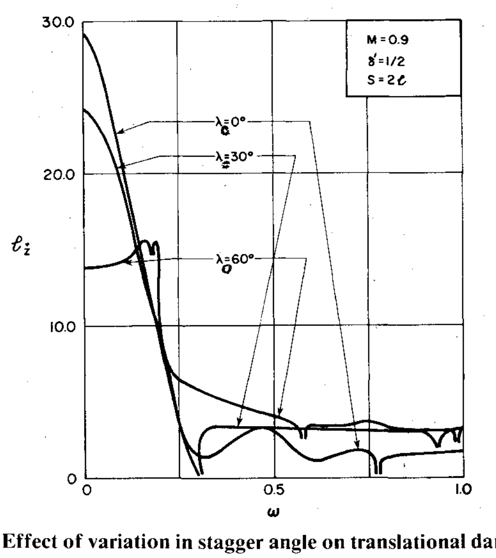


Fig. 6 Effect of variation in stagger angle on translational damping.

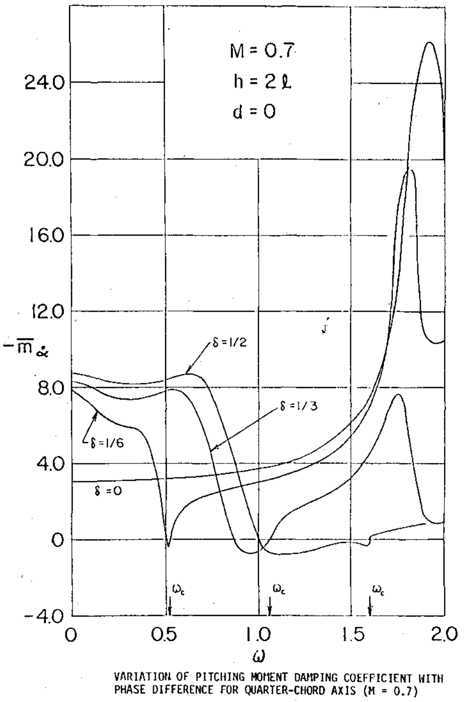


Fig. 7 Variation of pitching moment damping coefficient with phase difference for quarter-chord axis for $\lambda_0 = 0$ and $M = 0$.

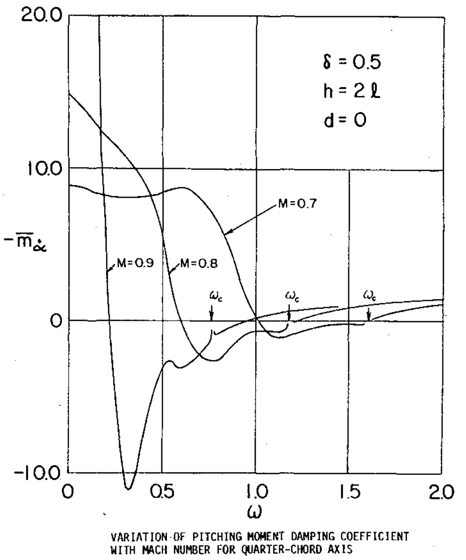
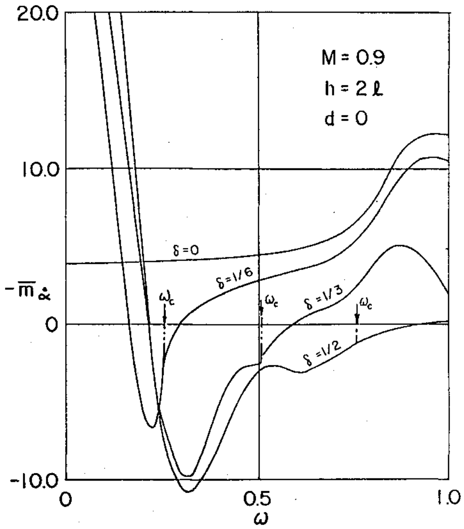


Fig. 8 Variation of pitching moment damping coefficient with Mach number for quarter-chord axis for $\lambda_0 = 0$.

For $M = 0.9$, the area of potential instability is extensive and the curve of $-\bar{m}_{\dot{\alpha}}$ shows a minimum value at about $\omega = 0.3$. The phase difference for this case was 180° and the airfoils were unstaggered. The effect of decreasing the phase difference is shown in Fig. 9. It reveals that a decrease in the phase difference would be beneficial. Figure 10 shows the variation in the damping $-\bar{m}_{\dot{\alpha}}$ when the airfoils are staggered and $\lambda_0 = 30°$. For the worst case considered, namely when $M = 0.9$ and $\delta' = ½$, Fig. 11 indicates that increasing the staggered angle would result in a reduction in the area of instability and would lessen the danger of flutter. It might, however, be worthwhile to carry out a full flutter analysis, using the appropriate aerodynamic coefficients, and comparing the results with those obtained by Whithead[18] by the use of Smith's method.[19]

Theoretical studies of blade vibration in axial compressors are often made by replacing the blades by a cascade of closely



VARIATION OF PITCHING MOMENT DAMPING WITH PHASE DIFFERENCE FOR QUARTER-CHORD AXIS (M = 0.9)

Fig. 9 Variation of pitching moment damping with phase difference for quarter-chord axis for $\lambda_0 = 0$ and $M = 0.9$.

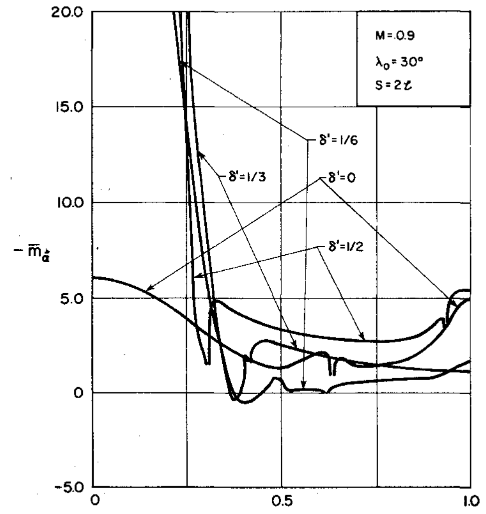


Fig. 10 Variation of pitching moment damping with phase lag for $\lambda_0 = 30°$.

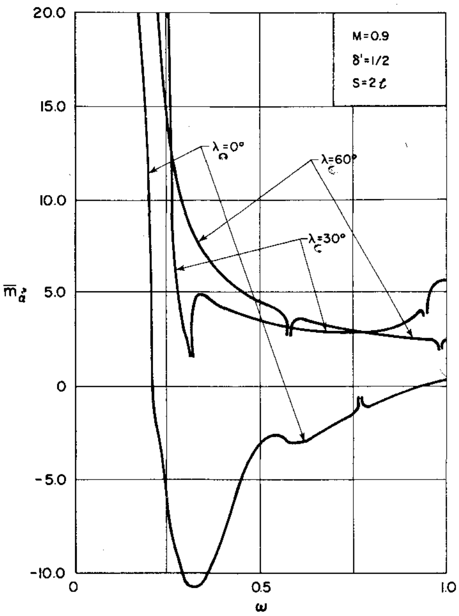


Fig. 11 Variation of pitching moment damping coefficient with stagger angle.

spaced airfoils. Such a representation by a two-dimensional cascade may be too idealized as in an actual compressor the speed of flow varies along the span of the blade. Blade tip effects and internal structural damping are also present that might tend to suppress theoretically predicted vibrations.

References

[1] Meister, E., "Flow of an Incompressible Fluid through an Oscillating Staggered Cascade" *Archives of Rational Mechanics and Analysis*, Vol. 6, 1960, pp. 198-230.

[2] Schorr, B. and Reddy, K. C., "Invisicid Flow through Cascades in Oscillatory and Distorted Flow," *AIAA Journal*, Vol. 9, Oct. 1971, pp. 2043-2050.

[3] Jones, W. P., "Wind-Tunnel Interference Effects on Measurements of Aerodynamic Coefficients for Oscillating Airfoils," British R&M 2786, Sept. 1950.

[4] Timman, R., "The Aerodynamic Forces on an Oscillating Airfoil between Two Parallel Walls," *Applied Scientific Research*, Vol. A3, No. 1, Feb. 1951, pp. 31-57.

[5] Lilley, G. M., "An Investigation of the Flexure-Torsion Flutter Characteristics of Airfoils in Cascade," The College of Aeronautics, Cranfield, Report 60, May 1952.

[6] Chang, C. C. and Chu, W. H., "Aerodynamic Interference of Cascade Blades in Synchronized Oscillation," *Journal of Applied Mechanics*, Vol. 22, 1955, pp. 503-508.

[7] Woods, L. C., "On Unsteady Flow Through a Cascade of Airfoils," *Proceedings of the Royal Society of London,* Series A228,50 (1955).

[8] Whitehead, D. S., "Force and Moment Coefficients for Vibrating Airfoils in Cascade," British R&M 3254, Feb. 1960.

[9] Runyan, H. L., Woolston, D. S., and Rainey, A. G., "Theoretical and Experimental Investigation of the Effect of Tunnel Walls on the Forces on an Oscillating Airfoil in Two-Dimensional Subsonic Compressible Flow," NACA T.N. 34156, June 1955.

[10] Jones, W. P., "Wind-Tunnel Wall Interference Effects on Oscillating Airfoils in Subsonic Flow," British R&M 2943, Dec. 1953.

[11] Fleeter, S., "Fluctuating Lift and Moment Coefficients for Cascaded Airfoils in a Non-Uniform Compressible Flow", *Journal of Aircraft,* Vol. 10, Feb. 1973, pp. 93-98.

[12] Rao, B. M. and Jones W. P., "Unsteady Airloads for a Cascade of Staggered Blades in Subsonic Flow", 46th Propulsion Energetics Review Meeting, Monterey, Calif., Sept. 1975.

[13] Jones, W. P. and Moore, J. A., "Aerodynamic Theory for a Cascade of Oscillating Airfoils in Subsonic Flow," Texas A&M University, TEES-3068-75-01 Feb. 1975 and "Aerodynamic Coefficients for a Staggered Cascade of Oscillating Airfoils in Subsonic Flow" Texas A&M University, TEES-3068-75-02, May 1975, College Station, Texas.

[14] Jones, W. P. and Moore, J. A., "Flow in the Wake of a Cascade of Oscillatory Airfoils," *AIAA Journal,* Vol. 10, Dec. 1972, pp. 1600-1605.

[15] Infeld, L., Smith, V.G., and Chien, W.Z., "On Some Series of Bessel Functions," *Journal of Mathematics and Physics,* Vol. XXVI, No. 1, April 1947, pp. 22-28.

[16] Kaji, S. and Okazaki, T., "Propagation of Sound Waves Through a Blade Row," *Journal of Sound and Vibration*, Vol. 11, No. 3, 1970, Parts I & II, pp. 339-375.

[17] Lane, F. and Friedman, M., "Theoretical Investigation of Subsonic Oscillatory Blade Row Aerodynamics", NACA, TN 4136, Feb. 1958.

[18] Whitehead, D. S., "The Effect of Compressibility on Unstalled Torsional Flutter", British A.R.C., R&M 3754, 1974.

[19] Smith, S.N., "Discrete Frequency Sound Generation in Axial Flow Turbomachines", British A.R.C., R&M 3709, March 1972.